Name: ....

Student Number: ....

Test 5 on WPPH16001.2020-2021
"Electricity and Magnetism"

Content: **4 questions with answers, grading scheme and analysis of typical mistakes**

Friday May 21 2021; **online**, 14:00-16:00

- Write your full name and student number on **each** page you use
- Read the questions carefully. Read them one more time after having answered them to ensure you have answered exactly what you were asked for.
- Compose your answers is such a way that it is well indicated which (sub)question they address
- Upload the answer to each question as a **separate pdf file**
- Do not use a red pen (it's used for grading)
- Griffiths' textbook, lecture notes and **your** tutorial notes are allowed. The internet, mobile phones, consulting, requests for consultancy and other teamwork are not allowed and considered as cheating

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov

Exam reviewed by (name second examiner) Steven Hoekstra

For administrative purposes; do NOT fill the table

The weighting of the questions:

	Maximum points	Averaged points scored
Question 1	10	6.5
Question 2	15	9.1
Question 3	11	7.3
Question 4	8	5.7
Total	**44**	**27.2**

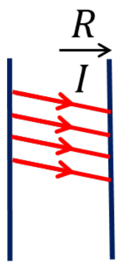
Grade = 1 + 9 x (score/max score).

Averaged grade: 7

Question 1. (10 points)

A long ideal solenoid with radius R and n turns per unit length, carries a constant current of I.

1. Find the energy density u stored in the fields. (2 points)



Now the current is slowly decreased to zero. The precise way of doing it doesn't matter; it could be e.g. a linear or exponential dependence on time.

2. Show that the electric field $\vec{\mathbf{E}}$ inside the solenoid in the quasistatic approximation is described as

$$\vec{\mathbf{E}} = -\frac{\mu_0 n}{2}\frac{dI}{dt} s\, \hat{\boldsymbol{\phi}} \quad \text{(2 points)}$$

where s is the distance from the solenoid axis in the cylindrical system of coordinates.

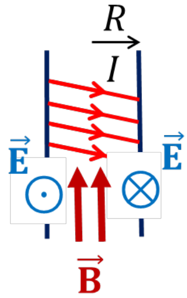
3. Calculate Poynting's vector inside the solenoid at the distance s from the solenoid axis. (2 points)

4. Show that the local energy conservation law holds. (4 points)

Answers

1. $\vec{\mathbf{B}} = \mu_0 n I\, \hat{\mathbf{z}}$ (inside; from the formula sheet)

Energy density: $u = \frac{1}{2}\frac{1}{\mu_0}B^2 = \frac{1}{2\mu_0}\mu_0^2 n^2 I^2 = \frac{\mu_0 n^2 I^2}{2}$ (2 points)



2. Using the Ampèrian loop as a circle with radius of s around the solenoid axis and applying Ampère's law in the integral form:

$$2\pi s E = \pi s^2 \frac{dB}{dt};\ \vec{\mathbf{E}} = -\frac{\mu_0 n}{2}\frac{dI}{dt} s\, \hat{\boldsymbol{\phi}} \quad \text{(2 points)}$$

NB: the direction of $\vec{\mathbf{E}}$ might be indicated in the figure; this is also correct.

UPD: $\frac{dI}{dt} < 0$ should be compensated by a minus sign in $\vec{\mathbf{E}}$, for it to be in positive $\hat{\boldsymbol{\phi}}$ direction. No points are deduced here for the wrong sign.

3. $\vec{\mathbf{S}} \equiv \frac{1}{\mu_0}\left(\vec{\mathbf{E}} \times \vec{\mathbf{B}}\right) = -\frac{1}{\mu_0}\frac{\mu_0 n}{2}\frac{dI}{dt}\mu_0 n I s\, \hat{\mathbf{s}} = -\frac{\mu_0 n^2}{2} I \frac{dI}{dt} s\, \hat{\mathbf{s}}$ (2 points)

Again, the sign of $\vec{\mathbf{S}}$ should be minus to be consistened with the right-hand system of coordinates. No points are deduced here for the wrong sign; only the correctness of expression counts.

4. Local energy conservation law:

$$-\frac{du}{dt} = \boldsymbol{\nabla} \cdot \vec{\mathbf{S}}; \quad \text{(1 point)}$$

Calculating $\boldsymbol{\nabla} \cdot \vec{\mathbf{S}}$ in cylindrical system:

$$\mu_0 n^2 I \frac{dI}{dt} = \frac{1}{s}\frac{\partial}{\partial s}\left(s \cdot s \frac{\mu_0 n^2}{2} I \frac{dI}{dt}\right) = \mu_0 n^2 I \frac{dI}{dt} \quad \text{(3 points)}$$

Holds!

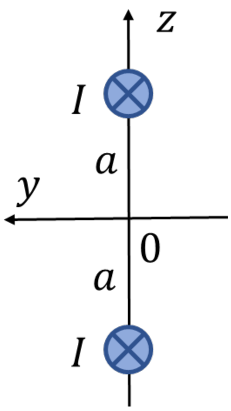
<u>Note for those who would like an extra challenge</u>: check if the Poynting theorem holds.

Typical mistakes:

- Vector notation was often neglected; especially in $\nabla \cdot \vec{\mathbf{S}}$
- Some students didn't realize $\vec{\mathbf{E}} = 0$ in question 1.1
- Some students didn't realize that $dW = 0$, as there are no charges present to exert work on
- Some students incorrectly applied the inverse chain rule; many forgot the factor ½ when contracting $I\frac{dI}{dt}$ into $\frac{dI^2}{dt}$.
- Some students tried finding the energy density using the inductance instead of using the formula relating them directly.
- Many students tried in 1.2 to explain a minus sign by noting that $\frac{dI}{dt} < 0$ is negative. This is not exactly a valid reasoning; the negativeness of $\frac{dI}{dt}$ is already in that expression implicitly. However, this was pardoned due to some inconsistency in the question.

Question 2 (15 points)

Consider two wires carrying identical currents I into the figure plane, separated by distance $2a$ as shown in the figure. The equidistant plane (i.e. where the distances between this plane and each wire in the set are equal) is the *xy* plane.



1. Find the magnetic field $\vec{\mathbf{B}}$ in the equidistant plane, expressed as a function of coordinate(s) and a. (4 points)

2. Show that the Maxwell stress tensor $\overleftrightarrow{\mathbf{T}}$ in the equidistant plane, expressed in the matrix form is given as:

$$\overleftrightarrow{\mathbf{T}} = \frac{\mu_0 I^2 y^2}{2\pi^2 (a^2 + y^2)^2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad \text{(5 points)}$$

3. Determine the force per unit length $\vec{\mathbf{f}}$ exerted on the lower wire by integrating the Maxwell stress tensor over the equidistant plane. Give your answers as $\vec{\mathbf{f}} = \cdots$ (6 points)

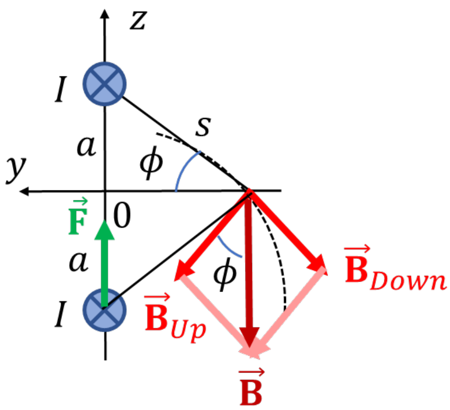
Tip 1: you might find useful to express the surface element $d\vec{\mathbf{a}}$ in the *xy* plane in the Cartesian (*not* cylindrical!) coordinates.

Tip 2: integrate over the y-coordinate from $-\infty$ to $+\infty$, and over the other coordinate from $-L/2$ to $+L/2$, and then divide the result on L

Tip 3: you might find useful the following integral:

$$\int_{-\infty}^{\infty} \frac{y^2}{(a^2 + y^2)^2} dy = \frac{\pi}{2a}$$

Answers (see similar Problem 8.4)



1. The magnetic field of an infinite wire is given in the extended formula sheet. The total magnetic field:

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_{Up} + \vec{\mathbf{B}}_{Down} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}_1 + \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}_2$$

$$= \frac{\mu_0 I \cos\phi}{\pi\sqrt{a^2+y^2}}\hat{\mathbf{z}}$$ (2 points)

because the horizontal components of the two fields cancel each other (how convenient!).

$$\cos\phi = \frac{y}{\sqrt{a^2+y^2}}$$ (1 point)

$$\vec{\mathbf{B}} = \frac{\mu_0 I y}{\pi(a^2+y^2)}\hat{\mathbf{z}}$$ (1 point)

Note that the magnetic field is independent of the x-coordinate, and has the z-component only.

UPD there was a typo in the sign of $\vec{\mathbf{B}}$: with the y-axis as shown, the sign should be plus as the y-direction is negative. No points are deduced for this.

2. As there is no electric field, Maxwell's tensor contains only the magnetic field components. Because the magnetic field has only the z-component, all non-diagonal elements are zero. (2 points)

$$T_{ij} \equiv \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$B^2 = \frac{\mu_0^2 I^2 y^2}{\pi^2(a^2+y^2)^2}$$ (1 point)

$$\overleftrightarrow{\mathbf{T}} = \frac{\mu_0 I^2 y^2}{2\pi^2(a^2+y^2)^2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$ (2 points)

3. Since we are in the magnetostatics regime, we know that the Poynting vector $\vec{\mathbf{S}} = 0$ (since there is no electric field). The force will only depend on the electromagnetic tensor and we calculate it in the following way:

$$\vec{\mathbf{F}} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0\mu_0 \frac{d}{dt}\int_V \vec{\mathbf{S}}\, d\tau = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\vec{\mathbf{a}}$$ (1 point)

$$\overleftrightarrow{\mathbf{T}} \cdot d\vec{\mathbf{a}} = \cdots \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}\begin{pmatrix} d\vec{\boldsymbol{a}}_x \\ d\vec{\boldsymbol{a}}_y \\ d\vec{\boldsymbol{a}}_z \end{pmatrix} = \cdots d\vec{\boldsymbol{a}}_z$$ (1 point)

so that F_z component of the force is non-zero and $F_x = F_y = 0$ (1 point)

In this particular case, we are evaluating the force exerted upon the lower wire. We are integrating other the xy-plane, therefore the only non-zero component of the infinitesimal surface vector is the one pointing in the z-direction $da_z = dx\, dy$:

$$F_z = \oint_S T_{zz} da_z = \frac{\mu_0 I^2}{2\pi^2}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int_{-\infty}^{\infty} \frac{y^2}{(a^2+y^2)^2} dy \quad \text{(1 point)}$$

$$= \frac{\mu_0 I^2}{2\pi^2} L \frac{\pi}{2a} = \frac{\mu_0 I^2}{2\pi} L \frac{1}{2a} \quad \text{(1 point)}$$

$$\vec{\mathbf{f}} = \frac{\mu_0 I^2}{4\pi a}\,\hat{\mathbf{z}} \quad \text{(1 point)}$$

Note for those interested: you can also verify this result by calculating the Lorentz force (Equation 5.16):

$$\vec{\mathbf{F}} = I\int \left(d\vec{\boldsymbol{l}} \times \vec{\mathbf{B}}\right) = IB_2 \int_{-L/2}^{L/2} dx\ \hat{\mathbf{z}} = IB_2 L\,\hat{\mathbf{z}}\,;\ \vec{\mathbf{f}} = \frac{\mu_0 I^2}{4\pi a}\,\hat{\mathbf{z}}$$

Typical mistakes:

- Some students appear to deduce the answer of 1.1 indirectly by looking at 1.2. However, as you are asked to derive the field, most points were granted for a correct derivation not your deduction abilities.
- Answers were occasionally given without direction.
- Incorrect integration of tensor. If you're unsure how it works, write everything out in vector and matrix form.

Question 3 (11 points)

An electromagnetic monochromatic plane wave with amplitude E_0, angular frequency ω and phase constant zero is traveling in the direction from the origin to the point (1, 1, 0), with polarization parallel to the xy plane.

1. Show that the explicit Cartesian components of $\vec{\mathbf{k}}$ and $\hat{\mathbf{n}}$ (i.e. the polarization vector) are

$$\vec{\mathbf{k}} = \frac{\omega}{c}\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}}\right) \text{ and } \hat{\mathbf{n}} = \frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}}{\sqrt{2}} \quad \text{(2 points)}$$

Tip: $|\vec{\mathbf{k}}|$ should be equal to $\frac{\omega}{c}$

2. Write down the (real) electric field $\vec{\mathbf{E}}(\vec{\mathbf{r}}, t)$. (4 points)

3. Write down the (real) magnetic field $\vec{\mathbf{B}}(\vec{\mathbf{r}}, t)$ for the same wave. Don't forget to express B_0 via E_0. (3 points)

4. Sketch the $\vec{\mathbf{E}}, \vec{\mathbf{B}}$ and $\vec{\mathbf{k}}$ vectors of this electromagnetic wave. (2 points)

Solution Question 3 (Griffiths 9.9 modified) (11 points)

1. $\vec{\mathbf{k}} = \frac{\omega}{c}\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}}\right)$; $\sqrt{2}$ comes from $|\vec{\mathbf{k}}| = \frac{\omega}{c} \rightarrow \frac{\omega}{c}\sqrt{\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}}\right)^2} = \frac{\omega}{c}\sqrt{\frac{1+1}{2}}$ (1 point for $\sqrt{2}$)

Since $\hat{\mathbf{n}}$ is parallel to the xy plane, it must have the form $\hat{\mathbf{n}} = \alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}}$; since $\hat{\mathbf{n}} \cdot \vec{\mathbf{k}} = 0, \beta = -\alpha$; and since $\hat{\mathbf{n}}$ is a unit vector, $\alpha = 1/\sqrt{2}$. (1 point)

$$\hat{\mathbf{n}} = \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}}{\sqrt{2}}$$

2. $\vec{\tilde{\mathbf{E}}}(\vec{\mathbf{r}}, t) = \tilde{E}_0 e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)}\,\hat{\mathbf{n}}$ (1 point)

NB: of course, using cos/sin notations is also correct.

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} = \frac{\omega}{c}\left(\frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}\right) \cdot (x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) = \frac{\omega}{\sqrt{2}c}(x + y) \quad \text{(2 points)}$$

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) = E_0 \cos\left[\frac{\omega}{\sqrt{2}c}(x + y) - \omega t\right]\left(\frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}}{\sqrt{2}}\right) \quad \text{(1 point)}$$

No vector signs - minus 1 point. Note that the directions might be (correctly) depicted in the figure below.

3. $\vec{\tilde{\mathbf{B}}}(\mathbf{r}, t) = \frac{\tilde{E}_0}{c} e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)}(\hat{\mathbf{k}} \times \hat{\mathbf{n}})$ because $B_0 = \frac{1}{c}E_0$ (1 point)

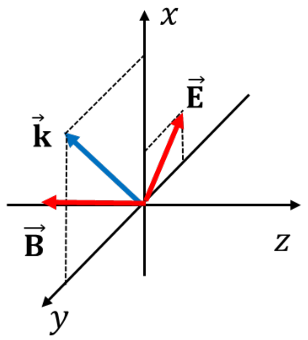
NB: of course, using cos/sin notations is also correct.

$$(\hat{\mathbf{k}} \times \hat{\mathbf{n}}) = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -\hat{\mathbf{z}} \quad \text{(1 point)}$$

$$\vec{\mathbf{B}}(\vec{\mathbf{r}}, t) = \frac{E_0}{c}\cos\left[\frac{\omega}{\sqrt{2}c}(x + y) - \omega t\right](-\hat{\mathbf{z}}) \quad \text{(1 point)}$$

No vector signs - minus 1 point. Note that the directions might be (correctly) depicted in the figure below.

4. (2 point in total)



Typical mistakes:

- Vector notation neglected.
- Some students did not use/derive the proper real field equation, and some did not compute at all the product between **k** and **n.**
- Some students had problems with the cross product computation, and some did not compute it at all.
- Majority of the students made a sketch of the wave, instead of the (more simple) vector sketch of the fields as was asked.
- Making drawings is apparently not a strong point of many…

Question 4 (8 points)

Imagine a space object with the mass of $m = 419000$ kg (similar to the International Space Station, ISS) and a cross-sectional area of $A = 2500\ \mathrm{m}^2$ (about the surface area of the solar panels on the ISS). Let's take the intensity of solar radiation as $I = 1.4\ \mathrm{kW/m^2}$ (similar to Earth's surface).

1. Calculate the electric field strength of the Sun radiation in this region in SI units. (2 points)

2. Calculate the force F_R on the space object due to solar radiation in SI units. Assume that all radiation is absorbed. (3 points)

3. Calculate the gravitational force

$$F_G = G\frac{Mm}{r^2}$$

of the Earth (gravitational constant $G = 6.67 \cdot 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$, Earth's mass $M = 5.9 \cdot 10^{24}$ kg) given that the object is approximately at $r = 10^8$ km from Earth's center. (2 points)

4. Which force is stronger? (1 point)

Solution Question 4

1. $I = \frac{1}{2}c\epsilon_0 E_0^2;\ E_0 = \sqrt{\frac{2I}{c\epsilon_0}}$ (1 point)

$$E_0 = \sqrt{\frac{2 \cdot 1.4 \cdot 10^3}{3 \cdot 10^8 \cdot 8.85 \cdot 10^{-12}}} \approx 10^3\ \frac{\mathrm{V}}{\mathrm{m}}$$ (1 point)

2. $p = \frac{I}{c} = \frac{1.4 \cdot 10^3}{3 \cdot 10^8} \approx 4.6 \cdot 10^{-6}\ \mathrm{N/m^2}$ (2 points)

$F_R = pA = 4.6 \cdot 10^{-6} \cdot 2500 = 1.2 \cdot 10^{-2}$ N (1 point)

3. $F_G = G\frac{Mm}{r^2} = \frac{6.67 \cdot 10^{-11} \cdot 5.9 \cdot 10^{24} \cdot 419000}{(10^{11})^2} = 1.6 \cdot 10^{-2}$ N (2 points)

4. $F_R \approx F_G$ (1 point)

NB. As some of you mentioned during the exam, the distance here is a bit too long. This is true indeed so you could think of reformulating this problem for more "realistic" settings (you might also consult the last-year exam). Pls end me you suggestions – to be used next year!

Typical mistakes:

- No units given
- Confusing E_0 with P=I/c
- In 4.3, some students also forgot to square the r in the denominator.
- The most mistakes resulted from simple calculation mistakes

Maxim Pchenitchnikov Steven Hoekstra

May 17 2021